1. Let $T : X \to Y$ be a linear map from a vector space $X$ to a vector space $Y$.

- Show that the kernel (null space) of $T$, $\ker T := \{x \in X : T(x) = 0\}$, and image (range) of $T$, $\operatorname{im} T := \{T(x) : x \in X\}$, are subspaces of $Y$ and $Y$, respectively.
- Starting with a basis $\{x_1, \ldots, x_k\}$ for $\ker T$, prove the dimension theorem; namely, if $\dim X < \infty$, then
  $$\dim X = \dim \ker T + \dim \operatorname{im} T.$$
- Prove that if $\dim X = \dim Y < \infty$, then the linear map $T$ is injective (one-one) if and only if it is surjective (onto).

2. Find a simple formula for $A^n$ for every positive integer $n$, where
  $$A := \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

3. Use the appropriate spectral theorems to prove each of the following:

- Let $U$ be an $n \times n$ complex matrix that is unitary, i.e. $UU^* = U^*U = I$, where $U^*$ is the conjugate transpose of $U$. Show that $U$ has a logarithm, i.e. there exists a matrix $V$ such that $e^V = U \iff V = \log U$.
- Let $A$ be a complex $n \times n$ matrix such that $A = A^*$ (i.e., the matrix is Hermitian), and let $\|A\|$ be that standard matrix norm induced by the standard inner product; namely,
  $$\|A\| := \max_{|x|=1} |Ax| = \max_{|x|=1} \sqrt{(Ax,Ax)} = \max_{|x|=1} \sqrt{(Ax)^T(Ax)}.$$
  Show that all eigenvalues $\lambda$ of $A$ satisfy $|\lambda| \leq \|A\|$ and that one or both of $+\|A\|$ or $-\|A\|$ is an eigenvalue of $A$.

4. The least squares approximation of an arbitrary polynomial $f(x)$ of degree $\leq n$ can be written as $r_n^*(x) = \sum_{j=0}^n a_j \phi_j(x)$, where $\phi_m(x), m \geq 0$, is an orthonormal family of polynomials with weight function $w(x) \geq 0$. Find $a_j, j = 0, \ldots, n$.

*Hint:* Define the inner product of two continuous functions $f$ and $g$ by
  $$(f, g) = \int_a^b w(x)f(x)g(x)dx, \quad f, g \in C[a,b],$$
  and the two norm by
  $$\|f\|_2 = \sqrt{\int_a^b w(x)[f(x)]^2dx}.$$
  Solve the least squares problem by minimizing $\|f - r\|_2^2$, where $r(x) = \sum_{j=0}^n a_j \phi_j(x)$.

(Over, please)
5. Consider the trapezoidal method,

\[ y(t_{i+1}) = y(t_i) + \frac{h}{2}(f(t_i, y(t_i)) + f(t_{i+1}, y(t_{i+1}))) \]

solved with one iteration using Euler’s method,

\[ y(t_{i+1}) = y(t_i) + hf(t_i, y(t_i)), i \geq 1, \]

as the predictor, for solving the IVP

\[ y' = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha. \]

- Form the difference equation.
- Compute the real part of the region of absolute stability for the method. (Hint: Use the model problem, \( y' = \lambda y \) with \( \text{Re}(\lambda) < 0 \) to find the values of \( \lambda h \) for which the method is stable.

6. (i) Consider the Jacobi

\[ x_i^{(k)} = \frac{1}{a_{ii}} \left[ b_i - \sum_{j=1, j \neq i}^n a_{ij}x_j^{(k-1)} \right], \quad i = 1, 2, \ldots, n \]

and Gauss-Seidel

\[ x_i^{(k)} = \frac{1}{a_{ii}} \left[ b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(k)} - \sum_{j=i+1}^n a_{ij}x_j^{(k-1)} \right], \quad i = 1, 2, \ldots, n \]

iterative methods for solving the linear system of equations \( Ax = b \). Represent the Jacobi and Gauss-Seidel iterations by \( x^{(k)} = T x^{(k-1)} + c \) and show that \( T = D^{-1}(\text{L} + \text{U}) \) and \( c = D^{-1}b \) for the Jacobi method and \( T = (D + \text{L})^{-1}(-\text{U}) \) and \( c = (D + \text{L})^{-1}b \) for the Gauss-Seidel method, where \( D \) is a diagonal matrix whose diagonal entries are those of \( A \), \( L \) is the strictly lower-triangular part of \( A \) and \( U \) is the strictly upper-triangular part of \( A \).

(ii) Show if \( \|T\| < 1 \), the Jacobi and Gauss-Seidel techniques converge and the following error bound holds

\[ \|x - x^{(k)}\| \leq \|T\|^k \|x^{(0)} - x\|, \quad x \in \mathbb{R}^n, \]

for any matrix norm.