1. (i) Suppose random variables $X$ and $Y$ have a joint distribution such that $E(Y|X) = X$. Show that, we must then have,
\[ \text{cov} (X,Y) = \text{var} X, \text{ and } \text{var} Y \geq 1. \]

(ii) Suppose $X \sim \text{Uniform} (0, 1)$. Let $a$ and $b$ be arbitrary fixed constants satisfying $0 < a < b < 1$. Consider the random variables,
\[ Y := 1_{\{0 < X < b\}}, \quad \text{and} \quad Z := 1_{\{a < X < 1\}}. \]

a) $Y$ ans $Z$ statistically independent? Justify your answer.

b) Are there value(s) of $a$, $b$ in $[0, 1]$, for which $Y$ and $Z$ are independent?

2. (i) Let $X_1, X_2, \cdots, X_n$ be a random sample of size $n \geq 2$ from a Normal distribution $N(\mu, \sigma^2)$. Let,
\[ \overline{X} = \frac{\sum_{i=1}^{n} X_i}{n}, \quad s^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \overline{X})^2 \]
be the sample mean and sample variance. Using moment generating functions, prove that
\[ \frac{(n-1)s^2}{\sigma^2} \sim \chi^2_{n-1}. \]

(Notation. $\sim$ means "distributed as". $\chi^2_k$ denotes a chi-square variable, with $k$ "degrees of freedom".)

You may use: (i) the additive property of chi-squares, (ii) $Z \sim N(0, 1) \implies \chi^2_1$, and (iii) independence of the sample mean and sample variance, when observations are from a Normal parent distribution.

(ii) Consider an exponential distribution (with a location parameter $\theta \in (-\infty, \infty)$, and scale parameter $\tau > 0$) having a density
\[ f(x) = \begin{cases} \tau^{-1} \exp\{-\frac{(x-\theta)}{\tau}\}, & \text{if } x > \theta \\ 0, & \text{otherwise} \end{cases} \]

Based on a random sample $X_1, X_2, \cdots, X_n$ of size $n \geq 2$ from this distribution, find a minimal sufficient statistic for $(\theta, \tau)$.
3. (i) Consider the exponential distribution in Problem 2(ii) above, with 
\( \tau = 1 \). Based on a random sample of size \( n \), find the MLE and the 
UMVU estimators of the location parameter \( \theta \). Compare the two 
estimators for unbiasedness, variance and consistency.

(ii) Consider a continuous distribution whose true density is \( f(x); \ x \in \ (-\infty, \infty) \). For the problem of testing
\[
H_0 : \quad f(x) = \pi^{-1/2} \exp (-x^2),
\]
versus
\[
H_1 : \quad f(x) = \pi^{-1}(1 + x^2)^{-1};
\]
construct the most powerful (MP) size-\( \alpha \) critical region, based on a 
single random sample \( X \), and compute the corresponding power.

4. (a) Let \( D \) be the differential operator on polynomials \( \phi(\omega) = \alpha_0 + \alpha_1 \omega + \alpha_2 \omega^2 + \alpha_3 \omega^3 + \alpha_4 \omega^4 \) of degree \( \leq 4 \). Thus \( D\phi(\omega) = \alpha_1 + 2\alpha_2 \omega + 3\alpha_3 \omega^2 + 4\alpha_4 \omega^3 \). Show that \( D \) is a linear operator:
\[
D (\alpha \phi(\omega) + \beta \psi(\omega)) = \alpha D\phi(\omega) + \beta D\psi(\omega)
\]
Given the isomorphism:
\[
\alpha_0 + \alpha_1 \omega + \ldots + \alpha_4 \omega^4 \sim col(\alpha_0, \alpha_1, \ldots, \alpha_4)
\]
find a matrix representation of \( D \).

(b) Let \( L \) be the space of vectors made up of real-valued functions 
\( \phi(t) \) at times \( t = 0, \pm 1/3, \pm 2/3, \ldots \) satisfying the condition of 
periodicity \( \phi(t + 1) = \phi(t) \). What is the dimension of \( L \)?

5. Let
\[
M = \begin{bmatrix} 3 & -2 \\ -2 & 6 \end{bmatrix}, \quad K = \begin{bmatrix} 39 & 2 \\ 2 & 36 \end{bmatrix}.
\]
Solve the generalized eigenvalue problem \( Kv_j = \lambda_j M v_j \) (\( j = 1, 2 \)), and 
find the matrix \( C \) such that \( C^* MC = I \) and \( C^* KC = diag(\lambda_1, \lambda_2) \). 
Given initial vectors \( x(0) \) and \( x'(0) \), show how the above results can 
be used to solve the initial-value problem:
\[
M x''(t) + K x(t) = 0
\]
6. (a) Write down the canonical Jordan matrix having eigenvalues \( \lambda = 2, 2, 1, 3, 3 \), one independent eigenvector for \( \lambda = 2 \), and two independent eigenvectors for \( \lambda = 3 \).

(b) Consider the matrix \( D = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) \) with eigenvectors \( u^1, u^2, \ldots, u^n \). If \( A = CDC^{-1} \) for some nonsingular matrix \( C \), how are the eigenvalues \( \mu_1, \mu_2, \ldots, \mu_n \) and eigenvectors \( v^1, v^2, \ldots, v^n \) of \( A \) related to those of \( D \)?